

Quantum field theory in quantum set algebra

David Ritz Finkelstein

Georgia Institute of Technology, Atlanta, Georgia*

finkelstein@gatech.edu

Abstract

A modular quantum architecture is given for the space-time, particles, and fields of the Standard Model and General Relativity. It assumes a right-handed neutrino, so that based on their multiplet structure all fundamental fermions have isospin $1/2$. This opens the possibility that the Higgs field can be identified with the Yang i -field of 1947. The quantum gravitational metric form proposed is a quantification of the Killing form of the quantum space-time cell. There is no trace of the black hole phenomenon at the one-cell quantum level.

1 On quantized space-time

This section title pays homage to Snyder and Yang [2, 3], who turned attention from the symmetry algebra of space-time to the dynamical algebra of a quantum in space-time, and regularized it in an attempt at a regular quantum field theory. Here the Yang regularization is extended from the abstract algebra to its representation, within the multiordinal Clifford algebra \mathcal{S} of [1], and a regular quantum field theory is outlined.

The canonically conjugate orbital variables x^m, p_m are derived here from basic Clifford spin variables γ^n much as Bose statistics is derived from odd statistics.

Yang [3] reformed the orbital algebra

$$\mathbf{a}_{\text{orb}} := \mathbf{a}(x^\mu, p_\mu, L_{\mu'\mu}, i) \quad (1)$$

*Emeritus.

of position, momentum, angular momentum, and $i = \sqrt{-1}$ to make it simple. Here all these operators have been made anti-Hermitian with factors of i where necessary. The reform of $\mathfrak{a}_{\text{orb}}$ is the algebra $\mathfrak{so}(5, 1)$ of angular momenta $J_{m'm}$ of a point particle in a space $6\mathbb{R}$ with dimensionless real coordinates η^m with invariant quadratic form

$$\begin{aligned} d\tau^2 &= g_{m'm} d\eta^{m'} d\eta^m, \\ g_{00} &= -1, \\ g_{11} &= g_{22} = g_{33} = g_{44} = g_{55} = +1, \\ J_{m'm} &= \eta_{m'} \partial_m - \eta_m \partial_{m'}. \end{aligned} \quad (2)$$

Introduce quantum units of time T and energy E , and adopt \hbar and c as units of action and speed. Yang's reformed orbital variables can be rewritten as

$$\begin{aligned} \overset{\circ}{x}^\mu &= \mathsf{T} J^{6\mu}, \\ \overset{\circ}{p}_\mu &= \mathsf{E} J_{5\mu}, \\ \overset{\circ}{L}_{\mu'\mu} &= J_{\mu'\mu}, \\ \overset{\circ}{i} &= \mathsf{E}\mathsf{T} J_{65}. \end{aligned} \quad (3)$$

Yang imposes a de Sitter subsidiary condition

$$[-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2]\psi = -\mathsf{E}^{-2}\psi. \quad (4)$$

The Yang x^1, x^2, x^3 again have (imaginary) discrete spectra but not x^0 .

To centralize $\overset{\circ}{i}$ requires that the angular momentum J be polarized so that J_{65} has a maximum magnitude j in the vacuum:

$$(J_{65})^2 \doteq -j^2, \quad j \gg 1. \quad (5)$$

The usual condition $i^2 = -1$ requires that

$$j\mathsf{E}\mathsf{T} = 1. \quad (6)$$

The Yang orbital commutation relations are a special case of the *simple quantization* relations

$$[q'', q^{n'}] = C^{n''n'}{}_n q^n \quad (7)$$

where C is the structure tensor of a simple Lie algebra \mathfrak{a} . In the Yang case $\mathfrak{a} = \mathfrak{so}(5, 1) \rightarrow \mathfrak{a}_{\text{orb}}$ and the q 's are 15 orbital variables. Canonical quantization is a singular limit of simple quantization as Bose-Einstein quantification is a singular limit of Palev quantification. Palev quantization is an inverse of the composite

process consisting of a Palev quantification with algebra \mathfrak{a} followed by a singular limit $J_{65}^2 \rightarrow \infty$ as the other components of J remain finite.

Thus the proposed simple quantum space-time is composed of finitely many spin-pairs of rank 5 having Palev statistics.

The general Palev quantization or quantification has the commutation relations (7) of a regular (semisimple, stable) Lie algebra. A Lie algebra that is semisimple but not simple arises when there are superselection laws between two subspaces of the port vector space. Such a limit to superposition implies that there are classical variables, so that the quantization process is probably not finished.

The orbital variables of the quantized space-time \mathcal{V} constructed next are again scaled components of an angular momentum tensor $J_{n'n}$ for $\mathfrak{spin}(3, 3)$ (or $\mathfrak{spin}(5, 1)$), but now that of the spinor space $\mathcal{S}(5)$ resulting from iterated quantification of the spin of $\mathcal{S}(3)$.

While $\mathcal{T}(2)$ has four Clifford generators γ^a , they have the neutral signature 0, not the Minkowski signature 2. To reform the orbital algebra $\mathfrak{a}_{\text{orb}} \rightarrow \mathfrak{spin}(3, 3)$ we need six Clifford units γ^y of neutral signature. $\mathcal{T}(3)$ is the lowest rank Clifford algebra in \mathcal{T} that will do, with 16 units γ^n of signature 0 and an angular momentum tensor $J_{n'n}(3) = \frac{1}{2}\gamma_{n'n}$ representing $\mathfrak{spin}(4, 4)$.

Yang chose the extra two dimensions to be spacelike for obvious reasons. Since the two extra dimensions are frozen, however, they can be timelike without disturbing macroscopic causality. $J_{n'n}$ represents on the particle rank 5 the angular momentum tensor $\gamma_{n'n}$ of the cell of rank 3.

The quantized space-time \mathcal{V} is constructed from six real γ^y ($y = 1, \dots, 6$) in $\text{Cliff}(3, 3) \cong \mathcal{T}(3)$. I commonly use a frame in which $g_{n'n}$ is diagonal and

$$\begin{aligned} g_{11} = g_{22} = g_{33} &= -g_{44} = -g_{55} = -g_{66} = 1, \\ \{\gamma^{n'}, \gamma^n\} &= 2g^{n'n}. \end{aligned} \tag{8}$$

Do not confuse this γ^5 with the usual γ^5 , now written γ^{4321} .

Represent the rank-3 orbital angular momentum tensor $\gamma_{y'y}(3)$ on rank-5 by double quantification:

$$J_{y'y}(5) = \sum_{(3)}^{(5)} \gamma_{y'y}(3). \tag{9}$$

Use $\hbar\mathsf{T}$ units, and set $\mathsf{E} = 1/\mathsf{N}$. Then the regular orbital variables in $\mathcal{T}(5)$ are

$$\begin{aligned} \overset{\circ}{p}_m &:= \frac{1}{\mathsf{N}} J^{m5} &:= \frac{1}{\mathsf{N}} \sum_{(3)}^{(5)} \gamma_{m5}, \\ \overset{\circ}{x}^m &:= J^{m6} &:= \sum_{(3)}^{(5)} \gamma^{m6}, \\ \overset{\circ}{i} &:= \frac{1}{\mathsf{N}} J^{65} &:= \frac{1}{\mathsf{N}} \sum_{(3)}^{(5)} \gamma^{65}, \\ \text{where } \overset{\circ}{J}_{m'm} &= -\overset{\circ}{J}_{mm'} &:= \sum_{(3)}^{(5)} \gamma_{m'm}, \end{aligned} \tag{10}$$

acting on $\mathcal{S}(5)$. The relation of T to the Planck time is still uncertain. N is the number of spins.

Now all variables that are diagonalizable have discrete bounded spectra. If a variable is antisymmetric, its square has a discrete bounded spectrum.

(10) stabilizes the Heisenberg Lie algebra and the Hilbert space:

$$\begin{aligned} \mathfrak{h}(x^m, p_m, J_{m'm}, i) &\hookleftarrow \mathfrak{spin}(3, 3) \\ &\cong \underbrace{\mathfrak{a}(\gamma^{21}, \dots, \gamma^{65})}_{15} \\ \mathcal{H} &\hookleftarrow \mathcal{S}(5). \end{aligned} \tag{11}$$

The canonical commutation relations between x and p emerge as the singular centralization $\mathsf{N} \rightarrow \text{inf ty}$, $\overset{\circ}{i} \rightarrow i$, a symmetry-reduction of the Clifford algebra $\text{op } \mathcal{S}(5)$. The complex Hilbert space of the canonical theory emerges from the real Grassmann algebra of the simplicial theory.

2 Green's functions

Scattering data of standard quantum field theory are found from Green's functions given by poorly defined integrals over field-histories,

$$\begin{aligned} G(x_1, \dots, x_n) &:= \frac{1}{\mathcal{N}} \int [d\psi] \exp iS \psi(x_1) \dots \psi(x_n), \\ \mathcal{N} &:= \int [d\psi] \exp iS \end{aligned} \tag{12}$$

where $\psi = (\psi(x))$ is a general field-history, $S = S[\psi]$ is the action functional operator of the field theory, and \mathcal{N} is a normalizing factor. Call the skew-symmetric iS the *skew action*. Reforming G is part of reforming the usual quantum field theory.

The singular integral over ψ was originally regarded as an integral over classical paths. There are no classical paths in Nature. Therefore the path integral is not an

operational description but a relict of the coarser and more singular classical theory. Quantum theory does not assign probability amplitudes to classical possibilities but determines new quantum possibilities too. It is easier to overlook this when the spectra of both the classical and quantum variables are continuous and the same.

The path integral is sometimes supposed to be a singular limit of a matrix product of many factors close to unity. But the actual factors in Nature seem to be portations of elementary quanta. These are very far from unity; their trace is 0 instead of large. One way to integrate over quantum histories composed of portations of odd quanta is to replace the formal integral \int in (12) by a Berezin integral $\int_{\mathbf{B}}$.

Operationally put, the usual Green's function (12) is the evaluation

$$G(x) = D \circ E(x) \quad (13)$$

of a history dynamics dual vector

$$D := \exp iS \quad (14)$$

on a history port vector $E = \Psi_n \dots \Psi_1$. D represents the system dynamics and E represents the quantum experiment. In \mathcal{S} models both are finite-dimensional and the contraction is regular. In standard gauge theories, non-compact gauge groups introduce new infinities. One point of this section is that in \mathcal{S} theories, such non-compactness is no problem.

It seems plausible that the Berezin integral in (12) becomes a finite trace in the operational formulation (13) when D and E belong to $\mathcal{S}(5)$. I accept this for now and proceed to reform the vectors D and $E(x)$ of the standard theory.

3 Dynamics

Consider only spinor fields $\psi(x)$. Then the Dirac skew action $iS[\psi]$ is an element of an infinite-dimensional Clifford algebra $\mathcal{C}_{\mathbf{D}}$. The linear operators on $\mathcal{S}(5)$ form a regular Clifford algebra $\mathring{\mathcal{C}}(5) = \mathcal{T}(5) = \text{op } \mathcal{S}(5)$. We may assume that the reform of the skew action functional iS is a polyadic $\mathring{S} \in \mathcal{T}(5)$.

The factor i in the exponent of $\exp iS$ requires special treatment. It is a singular limit of a quantification of a angular momentum γ_{65} , a component of the angular momentum tensor $\gamma_{y'y}$ in a regular Lie algebra $\mathring{\mathfrak{a}}$ that includes both the Yang orbital Lie algebra $\mathfrak{spin}(3, 3)$ and the regularization of the Standard Model internal Lie algebra \mathfrak{a}_{SM} of (22).

Hypothesis 1 (Regular dynamics) *There is a regular dynamics $\overset{\circ}{D} \rightarrow D$ before the singular limit $\overset{\circ}{i} \rightarrow i$, and it is invariant under $\overset{\circ}{\mathbf{a}}$.*

The singular limit \mathbf{a} includes x^μ among its generators. Therefore the Standard Model dynamics D is not invariant under \mathbf{a} . The singular limit must break $\overset{\circ}{\mathbf{a}}$ symmetry and leave at least the internal and orbital symmetries $\mathbf{a}_{\text{SM}} + \mathbf{a}_{\text{P}}$ of the Standard Model and the Poincaré group.

Then one immediate reformation replaces

$$\begin{aligned} i &\leftarrow \mathbf{N}^{-1} J_{m'm}, \\ S &\leftarrow \overset{\circ}{S}^{m'm}, \\ iS &\leftarrow \frac{1}{\mathbf{N}} \text{tr } J_{m'm} S^{m'm} \end{aligned} \quad (15)$$

where $\overset{\circ}{S}^{m'm}$ is a tensor dual to $J_{m'm}$ in its transformation. The polarization in this singular limit makes i dwarf the other components of $\mathbf{N}^{-1} J_{m'm}$ and reduces the trace to one term, the usual product.

The core of the usual dynamics is the Dirac skew action

$$A_{\text{D}} := iS_{\text{D}} := i \int [dx] \bar{\psi}(x) \gamma^\mu p_\mu \psi(x). \quad (16)$$

The Yang simplification makes the Poincaré-invariant Dirac operator $\gamma^\mu \partial_\mu$ a term in the Yang-invariant operator $\gamma^{y'y} J_{y'y}$. J has already been defined by quantification. It remains to define how the single-cell operator γ^y acts on the general multicell monadic. Every basic monadic is the association of a basic polyadic like those of Table I in [1]. A glance shows that every basic element has a tail of rank 3 on its right-hand side, representing a spinor of the Yang group. To act with any $\gamma \in \mathcal{T}(3)$ on a monadic e , remove the top bar of e , act with γ on its rank-3 spinor tail, and replace the bar. This operation can be written as

$$\gamma = \iota \gamma \bar{\iota}. \quad (17)$$

Here $\bar{\iota}$ is a left-inverse of ι and annihilates basic g -adics of grade $g \neq 1$:

$$\bar{\iota} \iota = 1, \quad \bar{\iota} \text{ grade} = \bar{\iota}. \quad (18)$$

And γ is extended from $\mathcal{S}(3)$ to \mathcal{S} by acting only on the rank-3 factor of its argument.

Following Yang, replace the vector factors and i in the Dirac action by angular momenta with two extra dimensions. The result is, up to normalization,

$$A \leftarrow \overset{\circ}{A} = J^{m''m'} \gamma_{m'}^m J_{mm''}, \quad (19)$$

an operator on $\mathcal{S}(5)$ of Clifford grade 6. Spinor indices of rank 3 on γ , of ranks 3 and 5 on J , and of rank 5 on A are unwritten. The Green's function depends on the signature chosen for the metric g but is not singular in any case.

Scattering port vectors must be defined before scattering amplitudes can be computed. The usual scattering ports determine four components of energy-momentum that do not commute in the \mathcal{S} theory. Coherent spin-states can be used instead [16].

4 Beneath dynamics

The question of the physical origin of dynamical law is often raised [10]. The dynamical law of the quantum system under study is violated at the beginning and end of every experiment, just as every symmetry is broken by the experimenter in order to be observed [11]. Here, as in the Dirac-Feynman summation over quantum histories, the dynamical law is summed up in a high-grade port vector D .

Every other port vector in physics represents portation to or from a statistical population in a reservoir outside the system. I must suppose that this holds for D too.

The dynamical “law” is then a statistical correlation between the system under study and its external source and sink. Since the ports are coherent (described by vectors rather than statistical operators) this correlation is an entanglement. Dynamics of a system is an entanglement with the rest of Nature.

5 Isospin and color

There are 16 kinds of first-generation fundamental fermion (briefly *ffermion*, pronounceable in Welsh) in the Standard Model, 4 leptons and 12 quarks; call them flavors.

Assume that a quark is a lepton with a color; that is, that there is a module $F = L + Q$ encoding the distinction among the ffermions, where the lepton tag L that makes a ffermion a lepton has grade 1, rank 3, multiplicity 4; the quark tag Q has grade > 1 , rank 3, multiplicity 12, and $Q = CL$, where C is color of multiplicity 3. It seems possible that the strong interactions exchange C 's, which leptons do not have. Leptons do have isospin in the form of $\text{so}(4\text{R})$, acting on the L in the lepton and the L in the quark.

Entire leptons suffer several independent dichotomies: τ_1 dividing left-handed electrons from left-handed neutrinos, chirality $i\gamma^{4321}$ dividing the left-handed leptons

from the right-handed, polarity Π dividing the lepton imports from the lepton exports, and particle number $N \doteq \pm 1$ dividing ffermions proper from anti-ffermions. These dichotomies cannot act independently on the lepton tag L alone. Chirality can act on the spin module of the ffermion; polarity can act on the entire ffermion. Then τ_1 and N act on the four leptons and also on the 12 quarks.

This classification fits nicely into $\mathcal{S}(3)$ if the ffermion *flavor module* has import space $\mathcal{F} = \mathcal{S}(3) \cong 16\mathbb{R}$. $\mathcal{S}(3)$ is spanned by the quantum sets with the four-place hyperbinary symbols for 0, 1, 2, ..., 15.

Its Clifford algebra has eight flavor generators $\gamma^f \in \mathcal{T}_1(3)$ ($f = 1, \dots, 8$). Take $\gamma^1, \dots, \gamma^4 \in \mathcal{T}_1(3)$ to be left-multiplications by $e^1, \dots, e^4 \in \mathcal{S}$ (see Table I in [1]), and $\gamma^5, \dots, \gamma^8$ to be the left partial differentiators with respect to the same four elements.

Set aside four generators γ^μ of the 16 in $\mathcal{T}(2)$ ($\mu = 1, 2, 3, 4$) to generate the Lorentz group; the four in $\mathcal{T}(1)$ having the wrong signature. Then extend them with $\gamma^5, \gamma^6 \in \mathcal{T}(2)$, both of the same signature, to six $\gamma^y \in \mathcal{T}(3)$ ($y = 1, \dots, 6$), to generate $\overset{\circ}{p}_\mu, \overset{\circ}{x}^\mu$, and $\gamma^{65} \sim \overset{\circ}{i} \rightarrow i$ as well.

Since left-handed and right-handed fermions have the same multiplicities, assume that both carry isospin 1/2 and that isospin commutes with parity P and chirality, unlike the Standard Model isospin.

Parity must still be violated. The isospin of the left-handed pro-particles and their anti-particles couple to the W meson; that of their mirror images does not. In \mathcal{S} models, this means that the W is a pair (or a number of pairs) of a left-handed odd pro-particle and its right-handed anti-particle. These pre- W 's are not necessarily able to exist as free particles, but since the W couples to both leptons and quarks the pre- W 's should be elements of both. This chiral isospin coupling correlation must appear in the dynamics vector responsible for the binding of fermions into a W , and ultimately in an alignment of the exosystem (the universe outside the system).

The Yang orbital Lie algebra $\mathbf{spin}(3, 3)$ and an internal Lie algebra $\mathbf{spin}(5, 5)$ are subalgebras of $\mathbf{spin}(5, 5)$. This differs from GUT $\mathbf{spin}(10)$ in signature.

The two values of isospin, Up and Down, may be attached to the first two binary places

$$|U\rangle = s_1 = \bar{1}, \quad |D\rangle = s_2 = \bar{=}, \quad (20)$$

which are indexed by their serial numbers in Table I. Similarly the three values of color may be identified with the three basis elements

$$|R\rangle = s_4 = \bar{=}, \quad |G\rangle = s_8 = \bar{=}\bar{=}, \quad |B\rangle = s_{12} = \bar{=}\bar{=}\bar{=}, \quad (21)$$

which span a color space $\mathcal{C} \subset \mathcal{S}$ such that $\mathcal{Q} = \mathcal{L} \wedge \mathcal{C}$ (quark = lepton \wedge color).

These five places support the Lie algebra $\mathbf{so}(5)$. This does not yet contain the GUT $\mathbf{so}(10)$; it must be doubled.

The Standard Model Lie algebra is

$$\mathfrak{a}_{\text{SM}} := \mathfrak{s}[\mathfrak{u}(2) \oplus \mathfrak{u}(3)] \quad (22)$$

composed of 5×5 matrices composed of diagonal blocks of 2×2 $\mathfrak{u}(2)$ matrices and 3×3 $\mathfrak{u}(3)$ matrices. While GUT theories simplify \mathfrak{a}_{SM} for the sake of unity, they also regularize it, eliminating hypercharge centrality. Neglecting signatures for the moment, the $\mathfrak{so}(10\mathbb{R})$ GUT regularization of the Standard Model and the Yang $\mathfrak{so}(6\mathbb{R})$ regularization of the Heisenberg-Poincaré group fit into one $\mathfrak{so}(16\mathbb{R})$ algebra acting on $\mathcal{S}(3)$.

6 Gauge hierarchy

Gauging a fiber Lie algebra \mathfrak{g} creates a bundle Lie algebra $\mathbf{G} = \mathfrak{g}^{\mathcal{M}}$ with a local isomorph of \mathfrak{g} at each point of Minkowski space-time \mathcal{M} , as in the Standard Model. The bundle algebra is not semisimple, being infinite-dimensional. Therefore it is reformed here.

Because gaugeons have even statistics, in an S theory they must be an even number of odd elements [17].

Quantification \bigwedge creates copies of the cells, each with a copy of the cell group. This suggests the

Hypothesis 2 (Gauge) *Gauging is quantifying a self-dual cell pair and taking a singular limit. The reform of the fiber gauge algebra is the cell Lie algebra. The reform of the bundle gauge algebra is the Lie algebra of the corresponding subspace of \mathcal{S} .*

The reform of the metric and the connection are taken up in Section 10.

This requires us to represent all the fiber groups of the Standard Model and gravity within the group of a cell pair. The least candidate for the port vector space of a cell is $\mathcal{S}(3)$, with 16 dimensions.

Roughly speaking, 6 dimensions of the 16 support the Yang orbital algebra $\mathfrak{spin}(6)$ and 10 support GUT $\mathfrak{spin}(10)$. More precisely, the signatures must fit within the neutral signature of $\mathcal{S}(3)$; and the two extra Yang dimensions of the 6 may participate in isospin.

The gauge (covariant) differentiator D is, up to a conventional constant $-i\hbar$, the total one-quantum momentum quantified, which in turn is a sector of the quantified

angular momentum

$$J(5) = \sum_{(3)}^{(5)} J(3). \quad (23)$$

Then the Lie algebra of the cell is

$$\begin{aligned} \overset{\circ}{\mathbf{G}} &= \text{grade}_2 \mathcal{T}(4) = \text{spin}(\text{dup } \mathcal{S}(3)), \\ \mathcal{T}(4) &= \text{op } \mathcal{S}(4) = \sqcap \text{dup } \iota' \mathcal{S}(3). \end{aligned} \quad (24)$$

$\mathcal{T}_2(4)$ has a Killing form $\mathbf{K}(4)$, a neutral quadratic form on $\mathcal{T}_2(4)$, invariant under the gauge group and unique in this regard up to a numerical factor.

Therefore I propose that this Killing form is the seed for the Einstein form of quantum gravity.

At the same time, this accounts for the hierarchy of the coupling constants. The Clifford units enter into the orbital momentum (10) with factors $1/\mathbf{N}$ and enter into the isospin and color with factors 1. The ratio of the coupling constants is then of order \mathbf{N} . For two electrons at atomic separations, the ratio of electric to gravitational interaction energy is $\sim 10^{42}$. This must be about the number of spins in an electron.

7 Higgs ϕ , Yang i

In the present script the deus ex machina of spontaneous organization is lowered twice, first with the Yang i and then with the Higgs ϕ_{H} . Perhaps cosmological inflation is also a spontaneous organization, resulting in another vacuum field, a Guth field ϕ_{G} . Can some of these three organizations be one?

i must have isospin 0 for the dynamics vector $D = \exp(iS)$ to be invariant under isospin $\text{spin}(3)$, and ϕ_{H} has isospin 1/2 in the Standard Model. It seems that they cannot be the same.

But it may soon be necessary to update the Standard Model. Isospins are historically assigned on the basis of multiplet structure, not interactions. If the right-handed neutrino exists then the left-handed and right-handed fermions all belong to doublets and must all be assigned isospin 1/2. I explore this possibility here.

If all the fermions have isospin 1/2, and the Higgs ϕ transforms as a fermion pair, it must now have isospin 0 or 1 instead of the previous value of 1/2. The term with isospin 1 is indispensable for distinguishing the electric from the weak interactions. If the pair is symmetric with respect to exchange of their isospin modules, the isospin 0 part vanishes. Then the Higgs ϕ has three real components, as in quaternion quantum mechanics [18], not two complex ones.

Parity violation is then concentrated in the interaction between the fermion and the W . In one parsimonious model W transforms as a self-dual pair of only a left-handed fermion module proper and its right-handed anti-fermion, without the opposite chiralities. It need not contain color modules.

Now an invariant dynamical vector D can be formed with an isovector $\overset{\circ}{i}$ in the form

$$D = \exp \overset{\circ}{i} \cdot S, \quad (25)$$

where both $\overset{\circ}{i}$ and S are isovectors, and the exponent is their scalar product.

To generate isospin $\mathbf{spin}(3\mathbb{R})$ within \mathcal{T} requires three Clifford monadics γ^i . To generate Yang $\mathbf{spin}(3,3)$ requires six γ^y . $i = \phi_H$ requires two of the γ^i to coincide with two of the γ^y . Take $y = 1, 2, 3, 4, 5, 6$ and $i = 5, 6, 7, 8$. This uses half of $\mathcal{T}(3)$

To represent $\mathbf{spin}(3\mathbb{R})$ in \mathcal{S} requires four Grassmann monadics e^5, e^6, e^7, e^8 , reserving e^1, e^2, e^3, e^4 for Lorentz $\mathbf{spin}(3,1)$. The isospin 3-vector is then

$$I_k = e^{k'} \tau_k \partial_{k'}, \quad \partial_k := \partial / \partial e^k. \quad (26)$$

As is well known, $\text{Cliff}(4,4) \cong \mathcal{T}(3)$ is a Botts period of the real neutral Clifford algebras, making it a mathematically natural module. It has a remarkable triality form

$$T = \bar{\psi} \gamma^n p_n \psi, \quad \psi \in \mathcal{S}_-(3), \quad p \in \mathcal{T}(2), \quad \bar{\psi} \in \mathcal{S}_+, \quad (27)$$

on the three isomorphic 8-dimensional vector spaces

$$\mathcal{S}_+(3) \cong \mathcal{T}(2) \cong \mathcal{S}_-(3); \quad (28)$$

meaning that the value of T on any non-zero vector in one of the three spaces is a duality between the remaining two spaces. Do these beautiful mathematical facts have physical meaning?

8 The 16 fermions

The dimension of $\mathcal{S}(r)$ grows hyperexponentially with rank r :

$$\begin{array}{rcccccc} \text{for } r & = & 0, & 1, & 2, & 3, & 4, & 5 \\ \dim \mathcal{S}(r) & = & 1, & 2, & 4, & 16, & 64\text{K}, & 2^{64\text{K}} \\ \dim \Delta \mathcal{S}(r) & = & 1, & 1, & 2, & 12, & 65520, & 2^{64\text{K}} - 64\text{K} \end{array} \quad (29)$$

Rank 5 is the first rank big enough for the orbital degrees of freedom of a quantum particle in a quasi-continuous space-time with chrone comparable to the Planck time. It is far oversize for the visible universe, but no smaller rank suffices.

If there is a dextral neutrino, there is a natural partition of the 16 kinds of Standard Model first-generation fundamental fermions that matches the partition of $\mathcal{S}(3)$ into four tiers $\Delta\mathcal{S}(r)$, $r = 0, 1, 2, 3$:

$$\begin{array}{c|ccccc}
r & 0 & 1 & 2 & 3 & \dots \\
\hline
\dim \Delta\mathcal{S}(r) & 1 & 1 & 2 & 12 & \dots \\
\text{SM fermions} & 1 & 1 & 2 & 12 &
\end{array} \tag{30}$$

This associates four leptons with tiers 0, 1, and 2, and 12 quarks with tier 3. These tiers do not contain the entire particle, which ranges up to tier 5 at least, but a module in the particle that determines its genus.

Since ranks 0, 1, 2, 3, 5, and 6 have plausible interpretations, probably rank 4 should have one too. Since it intervenes between quarks and free particles, one possibility to be considered is that rank 4 includes the confined assemblies like the nucleons. This hypothesis—if it lasts long enough to be dignified by that name—is vulnerable, since it must agree with the chromodynamic theory of confinement as far as that agrees with experiment, but I cannot think of another possible meaning in current experience for rank 4; perhaps it has none.

9 Generation

Generation-change preserves all the Standard Model symmetries but not the gravitational coupling through mass. Quantification preserves all the cell symmetries. This suggests the

Hypothesis 3 (Generation) *Generation is a rank to which gravitation couples.*

If all the fundamental quanta indeed have the same rank, this rank is not the rank of the particle itself but is presumably the rank of one of its lower-rank modules.

10 Quantum gravity

In the gauge theory of gravity, there is a covariant gauge differentiator D that serves as a potential for the gravitational gauge field $R := [D, D]$, the curvature. The gravitational potential D annuls the metric tensor g , $[D, g] = 0$, not by definition but as a constraint in the second-order theory of gravity, or as a dynamical equation in the first-order theory. Christoffel solved this for D in terms of g and its Lie derivatives ∂g , making g a potential for the potential. Call any dynamical field whose gauge derivative vanishes, a *metroid*, in analogy to the metric.

The Standard Model has a similar pattern of fields and potentials for the non-gravitational gauge fields. Let D include all the gauge potentials, from gravitational to chromodynamical, so that the gauge curvature R includes all the gauge fields. The metroids are the metric for gravity and the Higgs field for electricity.

In Dirac spin theory the local space-time metric is identified with the Dirac anticommutator

$$\{\gamma^{\mu'}, \gamma^{\mu}\} = 2g^{\mu'mu}. \quad (31)$$

This is a problem for quantum gravity: $g^{\mu'mu}$ is a classical quantity and cannot port gravitons. When the γ^{μ} belong to one cell, this g is an unlikely candidate for the seed of the gravitational

After Yang, infinitesimal space-time translations are associated with infinitesimal rotations $J_{y'y}$ in planes orthogonal to Minkowski space-time. The second-grade Clifford elements $\gamma_{m6}, \gamma_{m'6}$ in $\mathcal{T}(4) \cong \text{Cliff}(3, 3)$ represent quantum elements of momentum in the m and m' directions of the cellular quantum space-time \mathcal{Y} of Section 1. Their commutator is

$$[\gamma_{m5}, \gamma_{m'5}] = 2\gamma_{mm'}, \quad (32)$$

since $(\gamma^5)^2 = -1$. Since momentum generates translation, this commutator is an element of curvature in quantum units.

In classical differential geometry the commutator of two such small covariant momenta $P_{m'}, P_m$ would be the angular momentum $J_{m''m'''} R^{m''m'''}_{m'm}$, where R is the curvature tensor and J is a differential operator representing an infinitesimal orthogonal transformation. Here (32) becomes $\gamma_{m''m'''} R^{m''m'''}_{m'm}$.

According to (32), the default scale of magnitude of the effective R is two quantum units. If quantum units are comparable to Planck units, the quantum unit (qu) of curvature is enormous:

$$\begin{aligned} 1 \text{ qu}(\text{curvature}) &= 1 \text{ qu}(\text{length}^{-2}) \\ &\sim 1/(c\mathbb{T})^2 \\ &\sim 10^{70} \text{ m}^{-2}. \end{aligned} \quad (33)$$

Yet experimentally the local curvature is $\ll 1 \text{ m}^{-2}$. This means that at the quantum level of resolution R is a sum of many large terms of both signs that nearly average to 0, like the electric field in a conductor mapped at the resolution of the Bohr radius.

According to Riemann's inauguration lecture, continuous manifolds have no intrinsic metrical structure while discrete manifolds have an intrinsic metrical structure based on counting steps. Since quantum manifolds are neither quite continuous nor quite discrete, Riemann leaves us in a dilemma.

In fact a continuous group is a manifold but has intrinsic metrical structure, its Killing form. Since the orbital variables of a quantum theory generate a Lie group, they have its Killing form for a metric. This is therefore a candidate for the seed of the gravitational metric.

In theories of de Sitter and Yang, translations are approximations to rotations about a remote axis. Therefore they have two vector indices, not one. For the single cell with port algebra $\mathcal{S}(3)$, the orbital elements form the angular momentum tensor $\gamma_{m'm}$.

The Killing form is

$$K_{\{m'''m''\}\{m'm\}} := \text{tr } \Delta \gamma_{m'''m''} \Delta \gamma_{m'm}, \quad (34)$$

where Δ indicates the adjoint representation, the difference between left and right multiplication. This metric is related to curvature from birth. For an isolated small cell the center of rotation must be in the cell and the curvature of the Killing form is large.

K associates a number $K_{\{m'm\}\{m'm\}}$ with the orbital element $\gamma_{m'm}$ as classical metrics associate numbers with differentials of position. The quantum metric, however, should associate a graviton port, an operator in the Clifford algebra \mathcal{T} , with an orbital element. The number (34) could be its expectation value.

Formally, the trace in (34) is indeed an expectation value, namely that of the operator-on-operators

$$g^{\{m'''m''\}\{m'm\}} := \Delta \gamma_{m'''m''} \Delta \gamma_{m'm}. \quad (35)$$

This is therefore a better candidate for the seed of the quantum gravitational metric $\overset{\circ}{g}$ than K . Call this the Killing operator, since its trace is the Killing form. The quantized gravitational metric is then the quantification

$$\overset{\circ}{g}^{\{m'''m''\}\{m'm\}} = \mathbf{e} \Delta \gamma_{m'''m''} \Delta \gamma_{m'm} \bar{\mathbf{e}} \quad (36)$$

Like the $g^{\mu'\mu}$ of Special Relativity, this tensor is Lorentz invariant, in the same sense that the Dirac vector γ^m is when the written vector and unwritten spinor indices are consistently transformed. In the canonical theory $g_{\mu'\mu}$, like the tensor $p_{\mu'}p^{\mu}$, is symmetric and its components commute, but not so (36).

Regular gravitons cannot be exact bosons, since the Bose-Einstein commutation relations are singular, but may be palevons that usually pass for bosons. Then they should obey orthogonal-Lie-algebra commutation relations that have canonical ones as singular limit. In fact adjoint-formation Δ is a Lie-algebra isomorphism. The

$\Delta\gamma_{m'm}$ obey the same commutation relations as the $\gamma_{m'm}$. They therefore define the Palev statistics of an orthogonal Lie algebra.

At first glance the g of (35) does not look like an element of the Clifford algebra \mathcal{T} , as \mathcal{S} observables must be; but actually it is. Clifford algebras are closed under Δ : The left multiplications in g are in \mathcal{T} by definition, and the right multiplications can be expressed in terms of left multiplications and left differentiations, also in \mathcal{T} . This operator-valued form is then extended from a single cell to macroscopic orbital elements by multiple quantification.

Proposition 1 *When $\mathfrak{spin}(3,3)$ is reduced to the blocks of space-time x^μ , energy-momentum p_μ , Lorentz angular momentum $J_{\mu'\mu}$, and \mathbb{C} , the Killing form K on $\mathfrak{spin}(3,3)$ reduces to the direct sum of the usual classical physical metrics on these spaces.*

These are the Minkowski metrics on space-time and energy-momentum, the Lorentz Killing form $J^{\mu'\mu}J_{\mu\mu'}$ on the Lorentz angular momentum, and the usual $|z|^2$ norm on \mathbb{C} .

Proof. Direct calculation. The Killing form is diagonal in the basis $\gamma_{m'm}$, $m' > m$. \square

As a quick but rough check of this result, note that the signatures add up correctly:

$$2 + 2 + 0 - 1 = 3 = 9 - 6 \quad (37)$$

where the first 2 is the signature of Minkowski space-time, the second 2 is that of energy-momentum space, 0 is the signature of the Lorentz angular momentum $\|J\|$, -1 is the signature of \mathbb{C} , and $3 = 9 - 6$ is the signature of the Killing form of the 15-dimensional $\mathfrak{spin}(3,3)$.

If all its indices are different, $g_{\{m'''m''\}\{m'm\}}$ is symmetric under

$$\{m'''m''\} \leftrightarrow \{m'm\}. \quad (38)$$

This is not the case, however, for the momentum components of a single cell, where the fifth Yang dimension occurs twice among the four indices. Since the momentum p^μ is a quantification of $\gamma^{\mu 5}$ over \mathbf{N} cells, however, the pairs exchanged in (38) belong to different cells $\mathbf{N} - 1$ times more often than they belong to the same cell, and symmetry holds when $\mathbf{N} \rightarrow \infty$ even if some indices m repeat.

On the other hand, for finite large N , the physical case, there is a skewsymmetric part to g that is much smaller than the symmetric part, by a factor $\sim 1/\mathbf{N}$.

For decades Einstein studied the possibility that the electromagnetic field F is the skew-symmetric part \check{g} of the metric tensor. Here a skew-symmetric part \check{g} arises naturally in the quantum theory, which Einstein refused to consider, but disappears

in the classical limit. It would violate the Standard Model, in which F is part of the curvature $[D, D]$, not part of its second potential, to identify \check{g} with F . Furthermore the \check{g} field is as much weaker than gravity as the electric field F is stronger. I consider it no further.

11 The absence of Umklapp

A lattice space-time implies a cyclic energy-momentum space and Umklapp, a violation of energy-momentum conservation by the amount of the momentum period. \mathcal{S} quantum space-times like \mathcal{V} are not lattices but have non-commuting position variables and non-commuting momentum variables with bounded discrete spectra, neither periodic nor cyclic. They add like non-commutative angular momenta $J_{m'm}$, not commutative momenta. For example, a momentum componen p obeys

$$-j \leq J = p \leq +j. \quad (39)$$

There is therefore no Umklapp. The bounds have entropy 0, however, requiring temperature 0, and so are still experimentally unattainable, as when they were at infinity, but now they can be approached as closely as experimental resources allow, like absolute zero.

12 Summary

The indefinite Minkowski metric of space-time is a singular limit of a neutral modularized Yang space-time metric, which in turn derives from the Killing form of the basic cell.

Cartan used Grassmann algebra as unifying language for differential geometry but could not fit the symmetric metric into it, and so he introduced an ad hoc tensor product as well. His underlying manifold too is not defined today within a Grassmann algebra but within set theory.

In these \mathcal{S} models a multiordinal Grassmann algebra with its Clifford operator algebra serves as a universal language. The coordinate differentials are second-grade Clifford elements. Their metric form is now a fourth grade Clifford form, automatically symmetric under the exchange of two second-grade forms from different cells.

Now Grassmann algebra supplemented with the operator of association is the common language not only for the tangent structure but for the whole theory. For one cell, the Einstein metric tensor is the Killing metric tensor.

\mathcal{S} models based on Yang $\mathfrak{spin}(5, 1)$ or $\mathfrak{spin}(3, 3)$ predict a strong violation of the Heisenberg Indeterminacy Relations between position and momentum, allowing both their dispersions to be small at once, when $|J| \ll N$ and i is disorganized.

I tentatively associate rank 1, 2, and 3 of \mathcal{S} with the first generation neutrino space, lepton space, and fermion space of the Standard Model, each the Grassmann algebra of the preceding space; and the quantized i , with an extra spin operator J_{65} like Yang. This incorporates the three quark colors and 16 fermion flavors. The relation between chirality and isospin is determined by the structure of the vacuum. Space-time emerges from a coherent organization of the ranks from the spin rank 3 to the particle rank 5.

When the orbital Lie algebra $\mathfrak{a}_{\text{orb}}$ is reformed to $\mathfrak{spin}(3, 3)$ (or $\mathfrak{spin}(5, 1)$), all the orbital variables including i become components of the quantified angular momentum tensor $J = J_{y'y}$ of rank 3 ($y, y' = 1, \dots, 6$.) represented on rank 5.

The rank-4 fermion space $\mathcal{S}(4)$ contains the rank-3 lepton space $4\mathbb{R}$ and also three replicas of it, the quark $12\mathbb{R}$. That is why there are three colors.

Like set theory, the quantum architecture illustrated here is modular in the sense of Simon [6]. The spins $1/2$ in odd quantum theories naturally have various dimensionalities. This departs from the inspiring proposals of Penrose [5] and Feynman [4], and others, but brings their basic idea closer to the working quantum theories of today. Finite values for all observables is one immediate consequence: Clifford elements have finite spectra.

Because the Heisenberg indeterminacy principle is so weakened, it can no longer be excluded that gaugeons are pairs of odd quanta, though these odd quanta are not necessarily the ones able to exist as free quanta.

I have described a kinematics, but to compute masses and cross sections a more detailed dynamics must be set up and solved. Transition amplitudes of an odd quantum theory are traces of Clifford polynomials of huge degree, easy to write and hard to compute. Brute-force computation at the microscopic level for a macroscopic process is beyond the scope of any conceivable artificial computer. On the other hand, processes confined to some dozens of cells should be computable and might tell us something about very high energies; and some large but finite series can be summed in closed form.

Feynman was concerned by the Umklapp phenomenon that arises in a lattice space-time, with large violations of particle energy-momentum conservation. Umklapp happens because a spacial lattice breaks Poincaré (or Yang) invariance badly at momenta high enough to resolve the lattice. A quantum space-time with the Yang orbital algebra bends translational invariance slightly and has no Umklapp.

The quantization of i replaces the Heisenberg indeterminacy relation by the spin

indeterminacy relation. In principle the difference is observable and fixes the quantum of time.

13 Acknowledgments

I thank S. R. Finkelstein, H. Saller, and Frank “Tony” Smith, for discussions and information. Dialogues with Tenzin Gyatso helped me to shed the hypothesis of a permanent Natural Law [20]. The crucial importance of modularization dawned on me only after several of the annual Lindisfarne lectures of Lynn Margulis. Presentation of this work was aided by FQXi and the Templeton Foundation.

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